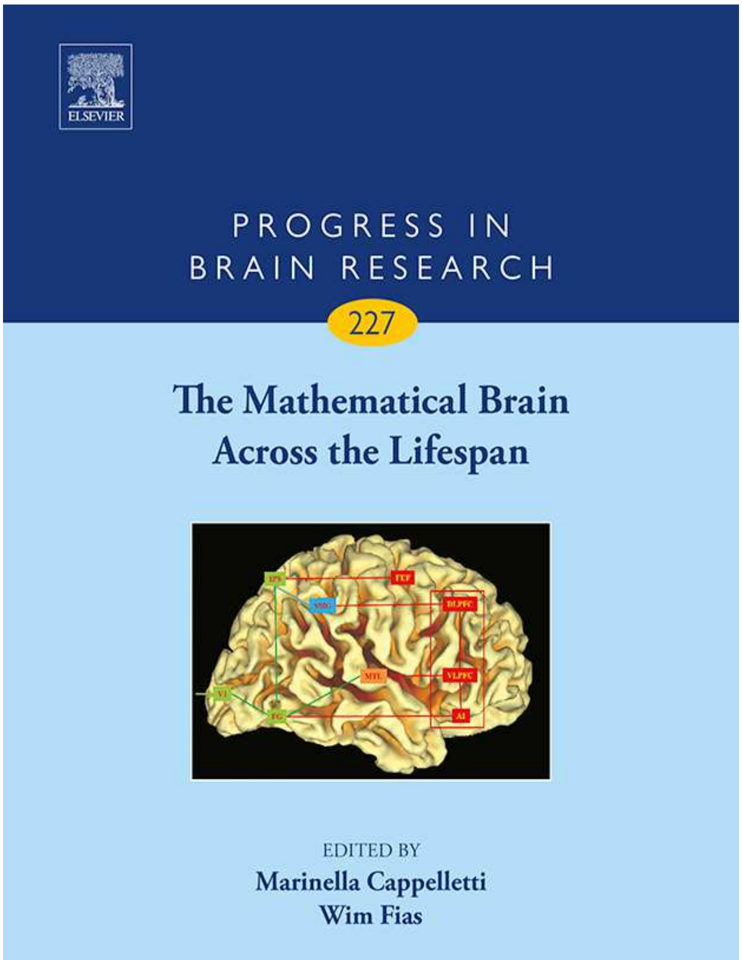

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CHAPTER 13

Approximate numerical abilities and mathematics: Insight from correlational and experimental training studies

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Abstract

Humans have the ability to nonverbally represent the approximate numerosity of sets of objects. The cognitive system that supports this ability, often referred to as the approximate number system (ANS), is present in early infancy and continues to develop in precision over the life span. It has been proposed that the ANS forms a foundation for uniquely human symbolic number and mathematics learning. Recent work has brought two types of evidence to bear on the relationship between the ANS and human mathematics: correlational studies showing individual differences in approximate numerical abilities correlate with individual differences in mathematics achievement and experimental studies showing enhancing effects of nonsymbolic approximate numerical training on exact, symbolic mathematical abilities. From this work, at least two accounts can be derived from these empirical data. It may be the case that the ANS and mathematics are related because the cognitive and brain processes responsible for representing numerical quantity in each format overlap, the Representational Overlap Hypothesis, or because of commonalities in the cognitive operations involved in mentally manipulating the representations of each format, the Operational Overlap hypothesis. The two hypotheses make distinct predictions for future work to test.

Keywords

Approximate number system, ANS, Number, Numerical cognition, Mathematics, Intraparietal sulcus, Parietal lobe, Training

[2]These authors made equal contributions to this work.

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1 COGNITIVE FOUNDATIONS FOR MATHEMATICAL ABILITIES

Only humans create and use symbolic number systems to carry out mathematics. Human use of symbolic number ranges from counting out change for monetary transactions to modeling the trajectory of spacecraft traveling beyond earth. As such, the ramifications of human creativity in symbolic number are broad in scope and profound in their cultural contribution. In modernized societies, number and mathematics are culturally transmitted through informal instruction and formal education. Children begin learning the meaning of number words before entering elementary school (see Carey, 2009) and continue to learn about numbers and mathematics, primarily through direct educational instruction, at least until early adulthood (if not beyond). However, several decades of developmental, cognitive, cross-cultural, and comparative psychology, as well as cognitive neuroscience, suggest numerical abilities precede direct instruction, education, or even spoken language (see Feigenson et al., 2004 for a review). More specifically, humans appear to be born with the capacity to mentally represent the approximate numerical magnitudes of sets (Izard et al., 2009; Xu and Spelke, 2000). This ability has been referred to by some as the "number sense" or approximate number system (ANS) (see Carey, 2009; Dehaene, 1997; Feigenson et al., 2004; Gallistel, 1990 for reviews). The ANS relies on a subset of posterior parietal regions of the brain within and around the intraparietal sulci (IPS) that selectively respond to number (Fig. 1; Cantlon et al., 2006; Dehaene et al., 2003; Piazza et al., 2004).

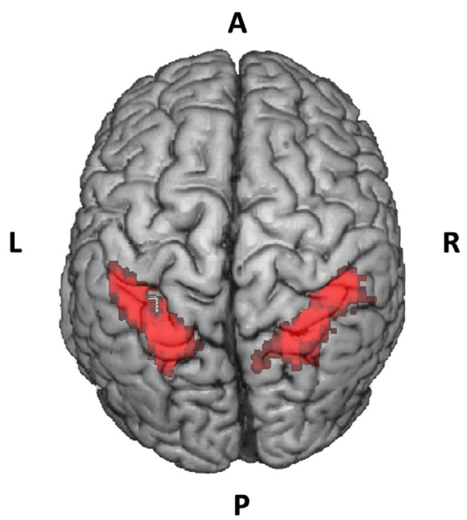

FIG. 1

Top-down view of bilateral parietal regions, including the intraparietal sulcus (IPS), implicated in numerical processing (L, left; R, right; A, anterior; P, posterior). Regions masked in *red* (*black* in the print version) were identified using http://www.neurosynth.org, an automated search of coordinates reported by archived publications (735 studies) using the term "number" and intersected with a parietal mask.

1.1 APPROXIMATE NUMBER SYSTEM

Neural and behavioral sensitivity to approximate numerosity is present from early infancy (Hyde and Spelke, 2011; Izard et al., 2009; Xu and Spelke, 2000). Within the first year of life, right intraparietal regions respond selectively to changes in the numerosity of visually presented sets of objects (Edwards et al., 2016; Hyde et al., 2010). Furthermore, young infants are behaviorally sensitive to the approximate numerical magnitudes of sets of objects (eg, Xu and Spelke, 2000). For example, after being repeatedly shown pictures containing the same number of items (eg, different pictures of 8 items) until boredom, 6-month-old infants become interested again, as indicated by an increase in visual attention, to a picture containing a different number of items (eg, 16 items; but remain bored if shown yet another novel picture containing the same number of items, 8) (Xu and Spelke, 2000). This reaction is based on the approximate number of items in the visual display, as changes in other nonnumerical variables like item size, spacing, and position are controlled between test pictures. The ANS increases in precision over development, where about a 1:2 ratio between numbers is needed to detect numerical differences in 6-month-old infants, but only a 7:8 to 10:11 ratio is needed in adults (see Halberda and Feigenson, 2008; Mou and vanMarle, 2014 for a review). Cross-cultural studies show that the ANS is universally shared across all humans, independent of language, cultural, or educational traditions, as remote peoples without formal numerical systems in their language or culture show evidence of nonverbal sensitivity to approximate numerosities (eg, Gordon, 2004; Pica et al., 2004). Finally, ANS does not even appear to be uniquely human, as many nonhuman animals also show the capacity to detect and act on approximate numerosity (Gallistel, 1990; Vallortigara et al., 2010).

A working theory in the field is that symbolic number and mathematical abilities arose, in part, from a foundation provided by the numerical capacities inherent in the ANS (Dehaene, 1997; Dehaene and Cohen, 2007; Feigenson et al., 2004; Gallistel and Gelman, 2000). Emerging evidence from correlational and longitudinal studies provides empirical evidence of a relationship between the ANS and symbolic number and mathematics (see Chen and Li, 2014; De Smedt et al., 2013; Fazio et al., 2014 for review). Recent experimental training studies, however, have begun to move beyond correlation to suggest the causal mechanisms underlying this relationship. These studies raise several possible hypotheses for future work to test.

1.2 ASSOCIATIONS BETWEEN APPROXIMATE NUMERICAL MAGNITUDES AND SYMBOLIC NUMBERS

Acquiring a symbolic number system requires us to learn the exact meaning of the number words or numerals (eg, “five” or “5” means exactly five items) (see Carey, 2009 for a review). Children growing up in modernized cultures typically learn these numerical meanings between the ages of 2 and 5, before entering formal school. At some point, we also come to associate approximate numerical meanings to number words and digits, and this association between the symbolic number system and ANS is evident in both brain and behavioral data of adults and older children. First, reaction time and accuracy in comparing two symbolic numbers (ie, digits or number words) is a function of their numerical distance or ratio from one another, just like it is for comparing nonsymbolic arrays of objects based on their numerosity, suggesting that their underlying meaning used to make the comparison in this context is approximate (eg, Moyer and Landauer, 1967; Pinel et al., 2001; Temple and Posner, 1998). Second, the speed of processing a symbolic number has been shown to be dependent on its numerical distance from a covertly primed number presented immediately beforehand (Van Opstal et al., 2008). Third, some overlapping number-selective regions of the IPS respond similarly to the numerical meaning of number words and digits, as well as the numerosity of sets of objects (eg, Piazza et al., 2007), suggesting that the systems are related on a neuroanatomical level as well as behaviorally.

1.3 CORRELATIONS BETWEEN APPROXIMATE NUMERICAL ABILITIES AND MATHEMATICS ACHIEVEMENT

A large number of studies have now shown a relationship between individual differences in ANS acuity and symbolic number and mathematics ability (see Chen and Li, 2014; De Smedt et al., 2013; Fazio et al., 2014 for review). In these studies, ANS acuity is typically obtained from measuring participants' performance in nonsymbolic numerical comparison tasks, where they are asked to judge which of two arrays of objects is more numerous (eg, Halberda et al., 2008). Many have now observed that the ability to approximately compare arrays of objects on the basis of number without counting is correlated with mathematics achievement scores in both children and adults (Halberda et al., 2008; Lourenco et al., 2012; Mazzocco et al., 2011; vanMarle et al., 2014). Many have shown that these correlations hold even after controlling for general cognitive abilities and linguistic abilities (Bonny and Lourenco, 2013; Halberda et al., 2008; Libertus et al., 2011; Mussolin et al., 2012; vanMarle et al., 2014). Furthermore, these correlations have been observed across the life span from early childhood (Bonny and Lourenco, 2013; Chu et al., 2013; Desoete et al., 2012; Libertus et al., 2011; Mussolin et al., 2012) to adulthood (Agrillo et al., 2013; DeWind and Brannon, 2012; Libertus et al., 2012; Lourenco et al., 2012; but see Inglis et al., 2011; Smets et al., 2014) and on a range of mathematics achievement assessments from preschool to the mathematics portion of college entrance exams.

Other studies have provided longitudinal data to show that approximate numerical abilities are predictive of later mathematics achievement (Chu et al., 2015; Libertus et al., 2013a). For example, Libertus et al. (2013a) found that 4-year-old children's ANS acuity is predictive of their math performance measured 6 months later, even controlling for working memory, attentional control, and vocabulary. The similar predictive effect of young children's ANS acuity on math performance is also observed in longitudinal studies with a larger time span (eg, 1 year in Chu et al., 2015; 2 years in Libertus et al., 2013b).

Even more strikingly, another recent study showed that individual differences in attention to approximate number at 6 months of age predict individual differences of mathematical abilities of the same children measured 3 years later, even after controlling for intelligence (Starr et al., 2013). This finding strongly suggests a scaffolding role for the nonverbal ANS in early symbolic mathematics learning.

1.4 EXPERIMENTAL TRAINING STUDIES ON THE RELATIONSHIP BETWEEN THE ANS AND MATHEMATICS

Recently, researchers have turned to experimental training studies in an attempt to better understand the causal mechanism(s) involved in the observed relationships between nonverbal, approximate numerical abilities with symbolic number and mathematics. This work has aimed at training the ANS directly and then measuring the influence of such training on symbolic number or mathematics performance. The logic of these studies is that if the ANS plays a casual role in a particular symbolic number or mathematics ability, then experimentally manipulating the ANS through cognitive training should influence symbolic number and mathematics performance relative to non-ANS training conditions. Work to date has focused on the effects of training the ANS on symbolic arithmetic (Hyde et al., 2014; Park and Brannon, 2013, 2014).

In a series of studies, Park and Brannon (2013, 2014) have shown that training the ANS enhances symbolic arithmetic abilities in adults. For example, in one study Park and Brannon (2013) asked adult participants to add and subtract arrays of dots approximately (without counting) for 10 sessions with pre- and posttesting on exact, multidigit arithmetic and vocabulary. More specifically, participants were asked to estimate the sum (add) of or the difference between (subtract) two dot arrays. For half the trials they were asked to determine if the answer they came up with was more or less than a third, foil array. In the other half of the trials, they had to choose which of two possible outcomes was the correct answer to the nonsymbolic arithmetic problem. Trial difficulty was manipulated over the course of training as participants became better. Change in performance from pretest to posttest on an arithmetic test and on a vocabulary test was compared between an experimental approximate arithmetic training group and an age-matched control group that did not engage any sort of training. Training participants showed greater improvement than control participants in nonsymbolic approximate arithmetic and exact, symbolic arithmetic tests, but not the vocabulary test. A second experiment reported that these training effects held against an active, yet nonnumerical, control training (trivia/fact training) and, further, the extent of improvement in approximate arithmetic correlated with the extent of improvement in symbolic arithmetic.

A more recent study by the same group (Park and Brannon, 2014) extended the training approach to systematically test several possible underlying mechanisms. They again randomly assigned adult participants to complete cognitive training in one of several conditions: approximate arithmetic, approximate numerical comparison, spatial working memory, and symbolic number ordering (Fig. 2). They found significant gains in symbolic arithmetic only for those who trained on approximate, nonsymbolic arithmetic. The fact that neither training on symbolic number ordering

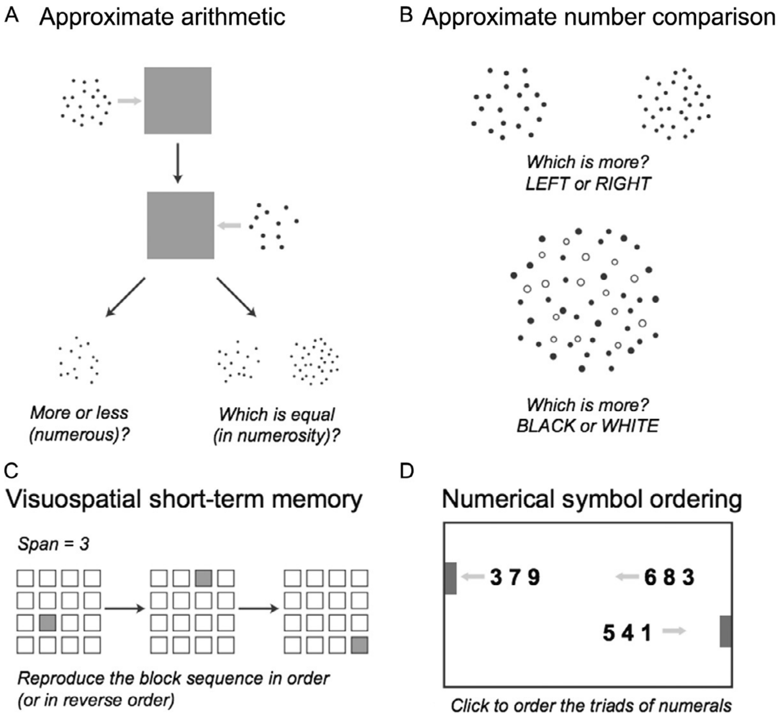

FIG. 2

Schematic depiction of training tasks used by Park and Brannon (2014).

Figure reprinted with permission from Park, J., Brannon, E.M., 2014. Improving arithmetic performance with number sense training: an investigation of underlying mechanism. Cognition 133 (1), 188–200.

nor training on general spatial working memory translated into symbolic arithmetic gains was interpreted as evidence that neither symbolic number automaticity nor working memory effects of ANS training alone can account for ANS training effects on symbolic arithmetic. Furthermore, the fact that gains in symbolic arithmetic were seen after approximate addition training but not after approximate numerical comparison training was interpreted as evidence that commonalities in the mental transformation of numerical representation between approximate and exact symbolic arithmetic, rather than engagement of the ANS alone, may be the driving mechanism behind ANS training effects on symbolic arithmetic performance.

Other experiments have shown that approximate number training enhances subsequent exact arithmetic performance in elementary school-aged children (Hyde et al., 2014). To do this, Hyde et al. (2014) asked first-grade children to practice one of several nonsymbolic, approximate magnitude tasks. The critical training condition, the approximate, nonsymbolic numerical addition condition, required children to estimate the sum of two sets of dots, which moved in turn, behind an occluder (Fig. 3). Children then had to indicate, by pressing a button, whether the

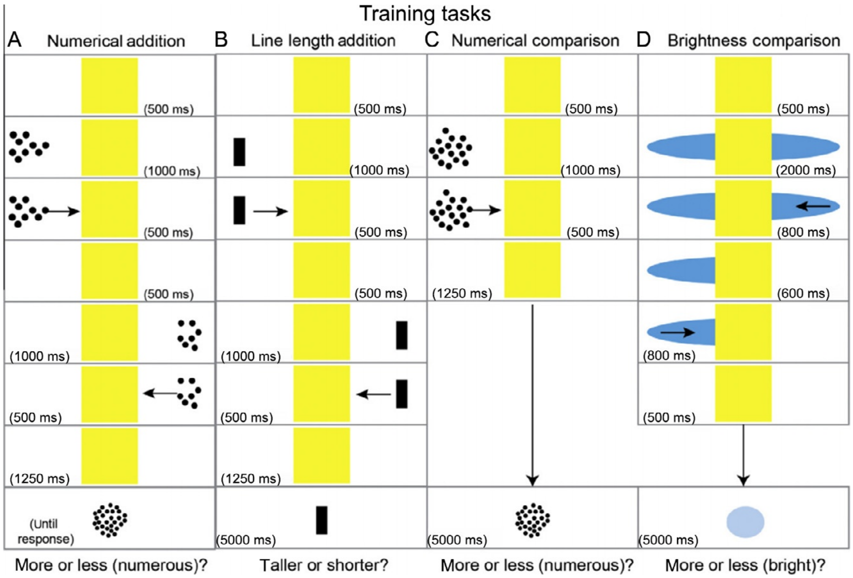

FIG. 3

Schematic depiction of approximate magnitude training tasks used by Hyde et al. (2014).

Reprinted with permission from Hyde, D.C., Khanum, S., Spelke, E.S., 2014. Brief non-symbolic, approximate number practice enhances subsequent exact symbolic arithmetic in children. Cognition 131 (1), 92–107.

number of dots revealed when the occluder disappeared was more or less than what they had anticipated the sum to be. Children practiced this task for about 15 min, after which they were given an exact, symbolic addition outcome test where they were asked to complete 1–3 digit symbolic addition problems on a paper worksheet. Over two experiments, test performance of the group of children assigned to the approximate, nonsymbolic numerical addition was compared to children assigned to other control training conditions involving nonnumerical magnitude addition and/or comparison with otherwise similar procedural and temporal demands as the approximate numerical addition condition. One control group, the line length addition group, was asked to mentally add lengths of lines disappearing behind an occluder and then compare the estimated sum of the two lines to the length of the line actually appearing when the occluder disappeared. Another group, the brightness comparison group, was asked to remember the brightness of an oblong object that moved behind the occluder, and judge whether the novel object that appeared when the occluder was removed was more or less bright than the original object. A final group, the approximate numerical comparison group, was asked to watch a single set of objects move behind an occluder and judge whether the array of objects revealed when the occluder was removed was more or less than the original set. In a first experiment, when symbolic addition test problems were relatively easy, those trained in the approximate numerical addition or the approximate numerical comparison condition completed the test problems significantly faster without any loss in accuracy compared to those that had practiced line length addition and those that had practiced brightness comparison. Additional analyses found that the effects of training condition on symbolic addition could not be explained more generally by accuracy or reaction time on the training task, suggesting that the content of training rather than particular performance or motivational aspects likely drove the effects.

These results suggest that brief practice with approximate number tasks can improve children's performance on a symbolic arithmetic test presented immediately afterward. The control conditions rule out a number of alternative accounts of the effects of ANS on symbolic number and mathematics. First, it does not seem to be the case that commonalities in the arithmetic operation between nonsymbolic addition training and symbolic addition can explain the results, as a group performing addition over line lengths did not perform as well as the approximate numerical addition group and the group performing approximate number comparison training performed equally well. Second, the enhancing effects of magnitude training appeared to be restricted to training with numerical magnitudes, as nonnumerical magnitude training (either brightness or line length) conditions matched for operation (addition or comparison) did not produce equal enhancements.

A follow-up experiment (Experiment 2, Hyde et al., 2014) compared new first-grade participants, trained on either the brightness comparison task or approximate numerical addition task, on more difficult 1–3 digit symbolic addition test problems and on a fill-in-the-blank sentence completion test. This experiment revealed that accuracy on symbolic addition was greater for those children trained on approximate numerical addition compared to those trained on brightness comparison. In contrast, no differences between groups were seen on the sentence completion task. These results suggest that ANS training can influence accuracy as well as speed. Furthermore, they suggest that the enhancing effects of ANS training are specific to symbolic number. The fact that training effects did not generalize to other nonnumerical tasks makes it unlikely that the mechanism driving training effects is a general increase in motivation or confidence, which would likely result in better overall performance on tests in a variety of cognitive and linguistic domains. Instead, these results suggest that the enhancing effects of ANS training were specific to the domain of number.

1.5 ALTERNATIVE EXPLANATIONS

There has been substantial debate in the literature regarding the nature of the actual number representation itself, hinging on the extent to which responses to nonsymbolic numerical arrays truly reflect abstract mental representations of number or whether such responses or judgments are better characterized as responses to nonnumerical sensory cues that often covary with or predict number (see Gebuis et al., 2014; Park et al., 2016). Since "number" must be extracted from sensory stimulation, it is uncontroversial to say that numerical perception and judgments are influenced by sensory stimulation. The extent of influence, however, is the matter of debate, with evidence ranging from responses being entirely dependently on nonnumerical aspects of stimulation (eg, Gebuis et al., 2014) to number being directly sensed (eg, Park et al., 2016; Ross and Burr, 2010). This debate, however, is largely irrelevant to the current argument, as regardless of which characterization of ANS one holds, behavioral and brain responses indicative of ANS engagement correlate with symbolic mathematics achievement and experimental training studies of the ANS show some evidence of transfer to mathematics.

It is also clear that performance on approximate numerical comparison tasks (and, by extension, approximate numerical addition tasks) requires more than the ANS. For example, the participant must, among other things, visually compare arrays, inhibit nonnumerical properties of the displays to extract number, hold numerosities in working memory, associate numerosities with side/spatial locations, and plan and execute motor responses to indicate which set has more. Recently, the issue of ANS task complexity has been raised in the literature and some have suggested that correlations between ANS performance and symbolic math may be explained by mediating general cognitive factors rather than by ANS acuity itself. As evidence of this, a number of correlational studies have now shown that controlling for executive functions severely reduces or even eliminates the relationship between the ANS and mathematics achievement (eg, Fuhs and McNeil, 2013; Gilmore et al., 2013; Inglis et al., 2011; Sasanguie et al., 2014, or see De Smedt et al., 2013 for a review). Furthermore, it seems as if different tasks, which are likely to have different nonnumerical task demands, give different and even sometimes unrelated estimates of ANS acuity (eg, Guillaume et al., 2016; Smets et al., 2014). What is clear from these studies is that general cognitive and motivational factors contribute to performance and should be considered as additional, if not major, sources of variance in performance on ANS acuity tasks.

While general cognitive factors like executive functions could mediate correlations between the ANS and mathematics achievement, it is less clear how the training effects reported earlier could be explained by mediating factors like executive function. For one, it is unlikely that training on ANS significantly increases executive function, especially given the time frame of training (ranging from a one, 15-min session to up to about a week and half at the most). Second, even if the ANS training was improving executive function, it is unclear how something like the inhibitory control required on the nonsymbolic ANS training task would directly transfer to gains on an exact, symbolic addition outcome measure. Third, control training conditions (like the nonnumerical magnitude training conditions of Hyde et al., 2014) are often matched on general cognitive task demands, yet do not transfer to gains in symbolic mathematics, suggesting that practice or training of the general cognitive abilities involved in the nonsymbolic approximate number tasks is not sufficient to cause gains in symbolic number and mathematics.

More broadly, it is uncontroversial to say that common general cognitive factors are involved in approximate number, symbolic number, and mathematics achievement tasks, as the same general cognitive factors are also involved in almost any task that can be construed as cognitive. Furthermore, it is highly likely that general cognitive factors, like executive functions, attention, and working memory, explain a large portion of variance in most cognitively demanding tasks. What has not yet been shown convincingly is that these factors account for all the variability predictive of mathematics achievement (see Schneider et al., 2016). As such, the idea that there are nonverbal number-specific processes that explain some portion of variance in symbolic number abilities should still be considered.

Another critique of training studies with active control conditions, such as most of the studies reported earlier, is that the expectations of participants in active control conditions are different from those in experimental training conditions, and it is this type of placebo-expectation effect, rather than actual training on particular types of content, that drives training effects (Boot et al., 2013). In the case of ANS training, it could be that participants form an expectation that they should be getting better at numerical tasks since they are doing a numerical training, and it is this placebo expectation that drives them to outperform on numerical posttests those participants training on nonnumerical tasks and who do not form the same expectations.

To investigate this issue, Dillon et al. (2015) questioned children about their expectations regarding the particular training conditions of Hyde et al. (2014). More specifically, they exposed children to a brief subset of practice problems from each training condition then, in turn, asked them how they thought practicing that particular task would influence their performance on the outcome measures of the study (ie, exact, symbolic addition). As a manipulation check to verify that children were carefully considering and coherently thinking about these questions, they also asked them how different environmental contexts (eg, a good nights sleep) would affect their performance on these tasks. They observed that children's expectations about how they would do on outcome measures did not align with the pattern of training enhancements actually seen in the study. More specifically, children thought practice on an approximate number task would improve approximate numerical abilities, but did not expect such practice to improve symbolic addition (the opposite pattern observed in the actual experimental data from Hyde et al., 2014). In contrast, they thought both a good night's sleep and eating their favorite breakfast would lead to enhanced performance on symbolic arithmetic. These results show that the expectations of children about the benefits of a particular ANS training (and other control conditions) do not align with the actual benefits observed in this ANS training study. These results suggest that ANS training effects are not likely due to placebo effects in experimental compared to active control conditions.

2 EMERGING IDEAS FROM EMPIRICAL WORK

At least two hypotheses can be derived from the empirical evidence to date to explain the relationship between the ANS and symbolic mathematics. One idea, which we will heretofore refer to as the *Operational Overlap Hypothesis*, is that commonalities in the cognitive processes involved in mentally manipulating symbolic and approximate number representations underlie associations between the ANS and symbolic

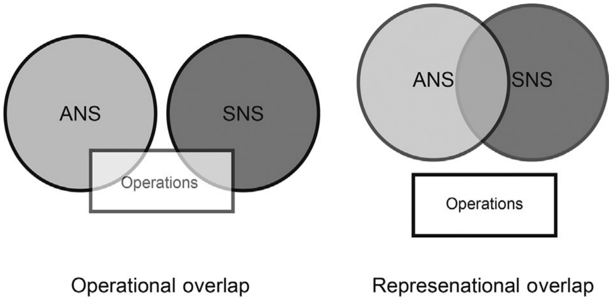

FIG. 4

Schematic diagram of emerging hypotheses to explain the relationship between the approximate number system (ANS) and symbolic number system (SNS). Item labeled "operations" represents the cognitive operations involved in mentally manipulating numerical representations.

number abilities (Fig. 4). A form of this idea has been previously proposed to explain cognitive training effects of approximate, nonsymbolic arithmetic on exact symbolic number arithmetic in adults (Park and Brannon, 2014). Under this view, the cognitive processes, including visual attention and working memory, employed specifically to do mental arithmetic over nonsymbolic arrays of objects and symbolic numbers, may be the same. As such, training mental manipulation of numerical representations in a nonsymbolic numerical context transfers to symbolic context. This idea explains results showing transfer of nonsymbolic approximate arithmetic training, but not nonsymbolic approximate numerical comparison, to exact symbolic arithmetic (Park and Brannon, 2014). Furthermore, this hypothesis can explain how individual differences in approximate numerical abilities are correlated with individual differences in symbolic number abilities in cases where common operations like arithmetic are required (eg, Gilmore et al., 2010).

It is unclear, however, how the *Operational Overlap Hypothesis* can explain correlations between ANS acuity itself and symbolic number or mathematics abilities that do not appear to share common operations or manipulations (Halberda et al., 2008; Libertus et al., 2011). Furthermore, it is unclear how this idea could explain the developmental training data, where both approximate number comparison training and approximate number addition enhance exact, symbolic addition (Hyde et al., 2014). The original constraint of this idea that the operations involved must be restricted to contexts like arithmetic where number representations are actually manipulated (Park and Brannon, 2014) could be relaxed to include other operations over mental representations like comparison to explain a wider variety of findings in the literature. However, even relaxing the definitions of the processes that constitute operational overlap does not account for developmental training results showing that practicing the same operations (comparison or addition) over nonnumerical approximate magnitude representations (brightness or line length) does not generalize to enhancements of exact, symbolic number addition.

A second idea, which we will heretofore refer to as the *Representational Overlap Hypothesis*, is that the ANS and symbolic number are related because the cognitive and brain systems associated with approximate number representation overlap to some degree with those associated with symbolic number representation (Fig. 4) (see Cordes et al., 2001; Dehaene, 1997; Gallistel and Gelman, 2000; Hyde et al., 2014; Piazza, 2010; Verguts and Fias, 2004 for similar ideas but with different characterizations of the ANS). Functional neuroimaging and neuropsychological work provides evidence that a subset of the regions that respond to nonsymbolic number also respond to symbolic number (eg, Dehaene et al., 2003; Edwards et al., 2016; Hyde et al., 2010; Piazza et al., 2007). Furthermore, approximate numerical magnitudes become associated with symbolic numbers over development (Lipton and Spelke, 2005; Temple and Posner, 1998) and the formation of such associations seems important for further mathematical understanding (Holloway and Ansari, 2009; Mussolin et al., 2012; Nys et al., 2013; vanMarle et al., 2014). The *Representational Overlap Hypothesis* predicts that the development or training of one system necessarily influences a subset of the mechanisms also involved in the other. This prediction is supported by the now extensive correlational work showing relationships between the ANS and symbolic mathematics achievement over the life span (eg, Halberda et al., 2008; Libertus et al., 2011, 2013a), as well as experimental studies showing ANS training in a variety of contexts can influence symbolic numerical performance and/or abilities (eg, Hyde et al., 2014).

Another prediction that follows from this hypothesis is that the relationship between the ANS and symbolic number will be bidirectional. Although most studies to date have examined the effects of approximate number on symbolic number, some initial evidence for bidirectionality has been observed. One cross-cultural study of the Mundurukú, an Amazonian hunter-gatherer tribe with extremely varied access to formal education, showed that education with symbolic numbers was associated with increased ANS precision (Piazza et al., 2013). More specifically, Mundurukú of all ages (4–60+) were tested on their approximate numerical comparison ability and their approximate size comparison ability, as well as surveyed on their educational experience (in years of schooling). Individuals with more education had more precise approximate numerical abilities, even after controlling for age. Furthermore, there was no effect of education on approximate size comparison abilities, suggesting that the relationship between education and comparison ability was specific to numerical magnitudes.

Several other studies have also reported effects of education on approximate number acuity in modern, industrialized samples as well (Guillaume et al., 2013; Lindskog et al., 2013; Nys et al., 2013). For example, educated individuals have more precise approximate numerical abilities than unschooled individuals (Nys et al., 2013), and college students in math-related disciplines have more precise approximate numerical abilities than those in nonmathematics-related disciplines (eg, Guillaume et al., 2013). In these cases, however, preexisting ANS differences cannot be ruled out. Others have shown that symbolic numerical abilities predict later performance on nonsymbolic comparison abilities after controlling for preexisting nonsymbolic numerical ability differences at the beginning of the study (Kolkman et al., 2013; Matejko and Ansari, 2016; Mussolin et al., 2014). Although this emerging evidence supports possible bidirectional effects predicted by the *Representational Overlap Hypothesis*, both the extent and symmetry of the overlap in the ANS and symbolic number are unspecified and are in need of further study.

Finally, without amendment the *Representational Overlap Hypothesis* cannot entirely explain the results of some training experiments with adults (Park and Brannon, 2014). According to this hypothesis, training of either approximate numerical comparison or approximate numerical addition should influence symbolic numerical and mathematical abilities. However, Park and Brannon (2014) found that only approximate numerical addition training, but not approximate numerical comparison training, led to gains in symbolic arithmetic. These results contrast with the results from children where both approximate addition and comparison practice led to gains in exact, symbolic addition (Hyde et al., 2014). Such differences in results between these training studies could reflect methodological differences in the nature and extent of training, rendering direct comparison between studies difficult. On the other hand, differences between training studies may reflect differences in the mechanisms driving relationship between the ANS and mathematics at different developmental time points. Further careful work is needed to determine which factors of ANS training are critical and how these factors might interact over development.

3 CONCLUSIONS

Several testable hypotheses with concrete predictions have been generated from experimental training studies to date, promising to reveal additional details regarding the underlying mechanisms driving correlations between the ANS and mathematics. Regardless of the underlying mechanisms, however, experimental training studies provide evidence that approximate number training can improve symbolic arithmetic in adults and children. As such, the methods of these training studies may also provide promise for translational work in mathematics education.

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